ÇANKAYA UNIVERSITY Department of Industrial Engineering
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IE 302 FACILITIES DESIGN AND LOCATION COURSE READING NOTES PART 2

TECHNIQUES FOR CONTINOUS SPACE LOCATION PROBLEMS

Continuous space location models determine the optimal location of one or more facilities on a two-dimensional plane. The obvious disadvantage is that the optimal location suggested by the model may not be a feasible one—for example, it may be in the middle of a water body, a river, lake, or sea. Or the optimal location may be in a community that prohibits such a facility. Despite this drawback, these models are very useful because they lend themselves to easy solution. Furthermore, if the optimal location is infeasible, techniques that find the nearest feasible and optimal locations are available.

We discussed the most important and widely used distance metrics: Euclidean, squared Euclidean, and rectilinear. We introduce three single-facility location models, each incorporating a different distance metric, along with the solution methods or algorithms for these models. Because the optimal solution for a continuous space model may be infeasible, where available, we also discuss techniques that enable us to find feasible and optimal locations.

Median Method

As the name implies, the median method finds the median location (defined later) and assigns the new facility to it. This method is used for single-facility location problems with rectilinear distance. Consider m facilities in a distribution network. Due to market- place reasons (e.g., increased customer demand), it is desired to add another facility to this network. The interaction between the new facility and existing ones is known. The problem is to locate the new facility to minimize the total interaction cost between each existing facility and the new one.

At the macro level, this problem arises, for example, when deciding where to locate a warehouse that is to receive goods from several plants with known locations. At the micro level, this problem arises when we have to add a new machine to an existing network of machines on the factory floor. Because the routing and volume of parts processed on the shop floor are known, the interaction (in number of trips) between the new machine and existing ones can be easily calculated. Other non-manufacturing applications of this model are given in Francis, McGinnis, and White (1992).

Consider this notation:

c_i cost of transportation between existing facility i and new facility, per unit

f_i traffic flow between existing facility i and new facility

$x_i y_i$ coordinates of existing facility i

The median location model is then to

$$\text{Minimize} \quad TC = \sum_{i=1}^{m} c_i f_i \left[|x_i - \bar{x}| + |y_i - \bar{y}| \right]$$

where TC is the total cost of distribution and $\bar{x}, \bar{y}$ are the optimal coordinates of the new facility.

Because the $c_i f_i$ product is known for each facility, it can be thought of as a weight w_i corresponding to facility i. In the rest of this chapter we therefore use the notation w_i instead of $c_i f_i$. We can rewrite expression (5) as follows:

$$\text{Minimize} \quad TC = \sum_{i=1}^{m} w_i \,|x_i - \bar{x}| + \sum_{i=1}^{m} w_i \,|y_i - \bar{y}| \qquad (6)$$

Because the x and y terms can be separated, we can solve for the optimal x and y coordinates independently. Here is the median method:

Median Method

Step 1 List the existing facilities in nondecreasing order of the x coordinates.

Step 2 Find the *j*th *x* coordinate in the list (created in step 1) at which the cumulative weight equals or exceeds half the total weight for the first time;

$$\sum_{i=1}^{j-1} w_i < \sum_{i=1}^{m} w_i / 2 \text{ and } \sum_{i=1}^{j} w_i \geq \sum_{i=1}^{m} w_i / 2$$

Step 3 List the existing facilities in nondecreasing order of the y coordinates.

Step 4 Find the *k*th y coordinate in the list (created in step 3) at which the cumulative weight equals or exceeds half the total weight for the first time:

$$\sum_{i=1}^{k-1} w_i < \sum_{i=1}^{m} w_i / 2 \text{ and } \sum_{i=1}^{k} w_i \geq \sum_{i=1}^{m} w_i / 2$$

The optimal location of the new facility is given by the *j*th x coordinate and the *k*th y coordinate identified in steps 2 and 4, respectively.

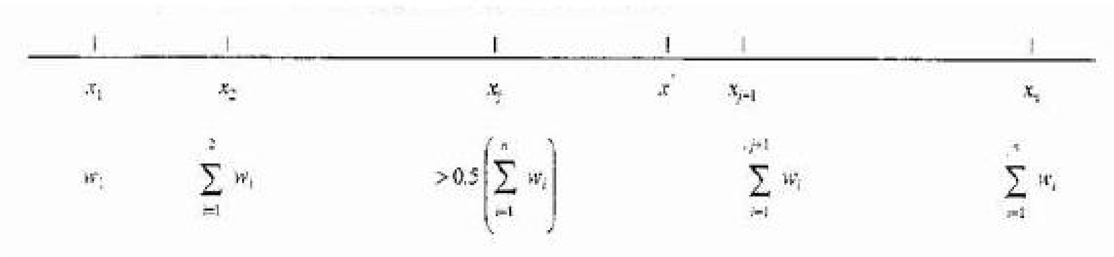
Four points about the model and algorithm are worth mentioning. First, the total movement cost—that is, the OFV of Equation (6)—is the sum of the movement costs in the x and y directions. These two cost functions are independent in the sense that the solution of one does not influence the solution of the other. Moreover, both cost functions have the same form. This means that we can solve the two functions separately using the same basic procedure, as we do in the median method.

Second, in step 2 the algorithm determines a point on the two-dimensional plane such that no more than half of the total traffic flow cost is to the left or right of the point. In step 4 the same is done so that no more than half of the total traffic flow cost is above or below the point. Thus the optimal location of the new facility is a median point.

Third, it can be shown that any other x or y coordinate will not be the same as the optimal location's coordinates; in other words, the median method is optimal. We offer an intuitive explanation. Because the problem can be decomposed into x axis and y axis problems and solved separately, let us examine the x axis problem—the following x axis movement cost function:

$$\sum_{i=1}^{m} w_i \,|\, x_i - \bar{x} \,|$$

Suppose the facilities are arranged in nondecreasing order of their x coordinates as shown in figure below. Let us assume that the x coordinate at which the cumulative weight exceeds half the total weight (for the first time) is the point shown as x_j in the figure. (The cumulative weights are shown below the respective coordinates in figure below. For coordinate x. we indicate that the cumulative weight exceeds half the total weight.) Let us also assume that the optimal x coordinate of the new facility falls at the coordinate indicated as x in the figure. For every unit distance we move to the left of x, the x axis movement cost decreases by more than half the total weight and increases by less than half the total weight. This is because the facilities to the left of x have a combined weight exceeding half the total weight and therefore those to the right of x (including x^*) must have a combined weight of less than half the total weight. Since every unit distance movement to the left improves the cost function, it is beneficial to keep moving to the left until we reach the x. coordinate. Any more movement to the left increases the total cost. Thus x_j must be the optimal coordinate for the new facility. In a similar manner we can establish the result for the optimal y coordinate.



Fourth, these coordinates could coincide with the x and y coordinates of two different existing facilities or possibly one existing facility. In the latter case, the new facility must be moved to another location because it cannot be located on top of an existing one! To determine alternative feasible and optimal locations, it is necessary to introduce the contour line concept. However, before we do this, we demonstrate the median method with an example.

Example 4

Two high-speed copiers are to be located on the fifth floor of an office complex that houses four departments of the Social Security Administration. The coordinates of the centroid of each department as well as the average number of trips made per day between each department and the copiers' yet-to-be-determined location are known and given in the following table

Department Number	x Coordinate	y Coordinate	Average Number of Daily Trips to Copiers
1	10	2	6
2	10	10	10
3	8	6	8
4	12	5	4

Assume that travel originates and ends at the centroid of each department. Determine the optimal location—the x, y coordinates—for the copiers.

Solution

We use the median method to get the solution.

Step 1

Department Number	x Coordinates in Nondecreasing order	Weights	Cumulative Weights
3	8	8	8
1	10	6	14
2	10	10	24
4	12	4	28

Step 2 Because the second *x* coordinate—namely, 10—in the list is where the cumulative weight equals half the total weight of 28/2 = 14, the optimal *x* coordinate is 10.

Step 3

Department Number	y Coordinates in Nondecreasing order	Weights	Cumulative Weights
1	2	6	6
4	5	4	10
3	6	8	18
2	10	10	28

Step 4 Because the third y coordinate in the above list is where the cumulative weight exceeds half the total weight of 28/2 = 14, the optimal coordinate is 6. Thus the optimal coordinates of the new facility are (10, 6).

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Although the median method is the most efficient algorithm for the rectilinear distance, single-facility location problem, we present another method for solving it that is used in the following chapters for the location of multiple facilities. It involves transforming the nonlinear, unconstrained model given by Equation (6) into an equivalent linear, constrained. Consider the following notation:

$$x_i^+ = \begin{cases} (x_i - \bar{x}) & \text{if } (x_i - \bar{x}) > 0 \\ 0 & \text{otherwise} \end{cases} \tag{9}$$

$$x_i^- = \begin{cases} (\bar{x} - x_i) & \text{if } (x_i - \bar{x}) \le 0 \\ 0 & \text{otherwise} \end{cases} \tag{10}$$

We can observe that

$$|x_i - \bar{x}| = x_i^+ + x_i^- \tag{11}$$
$$x_i - \bar{x} = x_i^+ - x_i^- \tag{12}$$

A similar definition of y_i^+, y_i^- yields

$$|y_i - \bar{y}| = y_i^+ + y_i^- \tag{13}$$
$$y_i - \bar{y} = y_i^+ - y_i^- \tag{14}$$

Thus the transformed linear model is:

$$\text{Minimize} \quad \sum_{i=1}^{n} w_i (x_i^+ + x_i^- + y_i^+ + y_i^-) \tag{15}$$

$$\begin{aligned} \text{Subject to} \quad & x_i - \bar{x} = x_i^+ - x_i^- & i = 1, 2, \ldots, n \quad & (11) \\ & y_i - \bar{y} = y_i^+ - y_i^- & i = 1, 2, \ldots, n \quad & (13) \\ & x_i^+, x_i^-, y_i^+, y_i^- \ge 0 & i = 1, 2, \ldots, n \quad & (16) \\ & \bar{x}, \bar{y} \text{ unrestricted in sign} & & (17) \end{aligned}$$

For this model to be equivalent.to (6), the solution must be such that either x_i^+ or x_i^-, but not both, is greater than zero. [If both are, then the values x_i^+ and x_i^- do not satisfy their definitions in (9) and (10).] Similarly, only one of y_i^+, y_i^- must be greater than zero. Recall that these conditions need to be satisfied for the LMIP models in Chapter 5, where we had to enforce them by introducing additional binary variables. Fortunately these conditions are automatically satisfied in the preceding linear model. This can be easily verified by contradiction. Assume that in the solution to the transformed model, x_i^+ and x_i^- take on values p and q, where p, q > 0. We can immediately observe that such a solution cannot be optimal because one can choose another set of values for x_i^+, x_i^- as follows:

$$x_i^+ = p - \min\{p, q\} \text{ and } x_i^- = q - \min\{p, q\} \qquad (18)$$

and obtain a feasible solution to the model that yields a lower objective value than before because the new x_i^+, x_i^-, values are less than their previously assumed values. Moreover, at least one of the new values of x_i^+ or x_i^- is zero according to the Expression (18). This means that the original set of values for x_i^+, x_i^-, could not have been optimal. Using a similar argument, we can show that either y_i^+ or y_i^- , will take on a value of zero in the optimal solution.

The model described by Expressions (11), (13), and (15)-(17), can be simplified by noting that x_i can be substituted as $\bar{x} - x_i^- + x_i^+$ from equality (12) and the fact that $\bar{x}$ is unrestricted in sign. Also y_i may be substituted similarly, resulting in a model with 2*n* fewer constraints and variables. Next we set up a constrained linear programming model for Example 4 and solve it using LINDO. The solution obtained, which has a total cost of 92, is the same as the one from the median method. Notice that XBAR, XPi, and XNi in the model stand for $\bar{x}, x_i^-, x_i^+$ respectively. Also, only one of XPi, XNi and YPi, YNi take on positive values. If XPi is positive in the optimal solution, it means that the new facility is to the left of existing facility i according to (9) and (10). Similarly, if YPi is positive, then the new facility is below existing facility *i*. Obviously, XBAR and YBAR give us the coordinates of the new facility's optimal location. As expected, we get the same solution obtained in Example 4.

```
MIN 6 XPI + 6 XM1 + 6 YPI + 6 YNI + 10 XP2 + 10 XN2 + 10 YP2 +10 YN2 + 8 XP3 + 8 XN3 + 8 YP3
+ 8 YN3 + 4 XP4 + 4 XN4 + 4 YP4 + 4 YN4

    SUBJECT TO
        2) XPI - XNI + XBAR = 10
        3) XP2 - XN2 + XBAR =10
        4) XP3 - XN3 + XBAR = 8
        5) XP4 - XN4 + XBAR = 12
        6) YPI - YNI + YBAR = 2
        7) YP2 - YN2 + YBAR = 10
        8) YP3 - YN3 + YBAR = 6
        9) YP4 - YN4 + YBAR = 5
    END
LP OPTIMUM FOUND AT STEP 11
    OBJECTIVE FUNCTION VALUE
    1) 92.00000
```

VARIABLE	VALUE	REDUCED COST
XPI	.000000	.000000

XNI	.000000	12.000000
YPI	.000000	12.000000
YM1	4.000000	.000000
XP2	.000000	12.000000
XN2	.000000	8.000000
YP2	4.000000	.000000
YN2	.000000	20.000000
XP3	.000000	16.000000
XN3	2.000000	.000000
YP3	.000000	8.000000
YN3	.000000	8.000000
XP4	2.000000	.000000
XN4	.000000	8.000000
YP4	.000000	8.000000
YN4	1.000000	.000000
XBAR	10.000000	.000000
YBAR	6.000000	.000000

ROW	SLACK OR SURPLUS	DUAL PRICES
2)	.000000	6.000000
3)	.000000	-2.000000
4)	.000000	-8.000000
5)	.000000	4.000000
6)	.000000	-6.000000
7)	.000000	10.000000
8)	.000000	.000000
9)	.000000	-4.000000

Contour Line Method

Suppose that the weight of facility 2 in Example 4 is increased to 20. Using the median method, we can verify that the optimal location's x, y coordinates are 10, 10. This location may not be feasible because it is department 2's centroid, and locating the two photocopiers in the middle of one department may not be acceptable to the others. We therefore wish to determine adjacent feasible locations that minimize the total cost function. To do so, we use the contour line method, which graphically constructs regions bounded by contour lines. Locating the new facility on any point along the contour line incurs the same total cost. Contour lines are important because if the optimal location determined is infeasible, we can move along the contour line and choose a feasible point that will have a similar cost. Also, if subjective factors need to be incorporated, we can use contour lines to move away from the optimal location determined by the median method to another point that better satisfies the subjective criteria.

We now provide an algorithm to construct contour lines, describe the steps, and illustrate with a numeric example.

Algorithm for Drawing Contour Lines

Step 1 Draw a vertical line through the x coordinate and a horizontal line through the y coordinate of each facility.

Step 2 Label each vertical line V_i, = 1, 2,..., p, and horizontal line H. ,j = 1,2,... ,q, where

V_i = sum of weights of facilities whose x coordinates fall on vertical line i
H_j = sum of weights of facilities whose y coordinates fall on horizontal line j

Step 3 Set i =j = 1 and $N_0 = D_0 = -\sum_{i-1}^{m} w_i$

Step 4 Set $N_i = N_{i-1} + 2V_i$ and $D_j=D_{j-1}+2H_j$. Increment i= i+ 1 and j =j + 1. If i ≤ p or j ≤ q, repeat 4. Otherwise, set i= j= 0.

Step 5 Determine Sij, the slope of the contour lines through the region bounded by vertical lines i and i + 1 and horizontal lines j and j + 1 using the equation Sij = -Ni, / Dj Increment i = i+ 1 and j =j + 1.

Step 6 If i ≤ p or j ≤ q, go to step 5. Otherwise, select any point (x, y) and draw a contour line with slope S_{ij}, in the region [i,j] in which (x, y) appears so that the line touches the boundary of this region. From one of the endpoints of this line, draw another contour line through the adjacent region with the corresponding slope. Repeat this until you get a contour line ending at point (x, y). You now have a region bounded by contour lines with (x, y) on the boundary of the region.

We discuss four points about this algorithm. First, the numbers of vertical and horizontal lines need not be equal. Two facilities may have the same x coordinate but not the same y coordinate, thereby requiring one horizontal line and two vertical lines. In fact, this is why the index i of V_i ranges from one top and that of H_j ranges from one to q.

Second, the N_i and D_j computed in steps 3 and 4 correspond to the numerator and denominator, respectively, of the slope equation of any contour line through the region bounded by the vertical lines i and i + 1 and the horizontal lines j and j + 1. To verify this, consider the objective function (6) when the new facility is located at some point (x, y)—that is, $x = x$, $y = y$:

$$TC = \sum_{i=1}^{m} w_i \mid x_i - x \mid + \sum_{i=1}^{m} w_i \mid y_i - y \mid \qquad (19)$$

By noting that the V_i's and H_j's calculated in step 2 of the algorithm correspond to the sum of the weights of facilities whose *x, y* coordinates are equal to the *x, y* coordinates, respectively, of the ith, jth distinct lines and that we have *p, q* such coordinates or lines *(p*

< m, q <. m), we can rewrite (19) as follows:

$$TC = \sum_{i=1}^{p} V_i \mid x_i - x \mid + \sum_{i=1}^{q} H_i \mid y_i - y \mid \qquad (20)$$

Suppose that *x is* between the sth and (s + 1)th (distinct) *x* coordinates or vertical lines (since we have drawn vertical lines through these coordinates in step 1). Similarly, let y be between the *t*th and (*t* + 1)th vertical lines. Then

$$TC = \sum_{i=1}^{s} V_i (x - x_i) + \sum_{i=s+1}^{p} V_i (x_i - x) + \sum_{i=1}^{t} H_i (y - y_i) + \sum_{i=t+1}^{q} H_i (y_i - y) \qquad (21)$$

Rearranging the variable and constant terms in Equation (21) we get

$$TC = \left[\sum_{i=1}^{s} V_i - \sum_{i=s+1}^{p} V_i \right] x + \left[\sum_{i=1}^{t} H_i - \sum_{i=t+1}^{q} H_i \right] y - \sum_{i=1}^{s} V_i x_i$$
$$+ \sum_{i=s+1}^{p} V_i x_i - \sum_{i=1}^{t} H_i y_i + \sum_{i=t+1}^{q} H_i y_i \qquad (22)$$

The last four terms in Equation (22) are constants. To make our discussion simpler, we substitute another constant term c. Also, the coefficients of x can be rewritten as follows:

$$\sum_{i=1}^{s} V_i - \sum_{i=s+1}^{p} V_i - \sum_{i=1}^{s} V_i + \sum_{i=1}^{s} V_i \qquad (23)$$

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Notice that all we have done in (23) is added and subtracted sum of Vi to the original coefficient. Because it is clear from step 2 that

$$\sum_{i=1}^{p} V_i = \sum_{i=1}^{m} w_i$$

the coefficient of x from (23) can be rewritten as:

$$2\sum_{i=1}^{s} V_i - \left[\sum_{i=1}^{s} V_i + \sum_{i=s+1}^{p} V_i\right] = 2\sum_{i=1}^{s} V_i - \sum_{i=1}^{p} V_i = 2\sum_{i=1}^{s} V_i - \sum_{i=1}^{m} w_i \qquad (24)$$

Similarly, the coefficient of y is

$$2\sum_{i=1}^{t} H_i - \sum_{i=1}^{m} w_i \qquad (25)$$

Thus Equation (22) can be rewritten as:

$$TC = \left[2\sum_{i=1}^{s} V_i - \sum_{i=1}^{m} w_i\right]x + \left[2\sum_{i=1}^{t} H_i - \sum_{i=1}^{m} w_i\right]y + c \qquad (26)$$

The N_i computation in step 4 is in fact this calculation of th coefficient if x. To verify this, note that $N_i=N_{i-1}+2V_i$. Making the substitution for N_{i-1}, we get $N_i=N_{i-2} + 2V_{i-1} + 2V_i$. Repeating this procedure of making substitutions for N_{i-2}, N_{i-1},... we get

$$N_i = N_0 + 2V_1 + 2V_2 + \cdots + 2V_{i-1} + 2V_i = -\sum_{i=1}^{m} w_i + 2\sum_{k=1}^{i} V_k \qquad (27)$$

Similarly, the reader can verify that

$$D_i = -\sum_{i=1}^{m} w_i + 2\sum_{k=1}^{i} H_i \qquad (28)$$

Hence Equation (26) is

$$TC = N_s x + D_t y + c,$$

which can be written as

$$y = -\frac{N_s}{D_t}x + (TC - c) \qquad (29)$$

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This expression for the total cost function at *x, y* or, in fact, any other point in the region *[s, t]* has the form $y = mx + c$, where the slope $m = -N/D_t$. This is exactly how the slopes are computed in step 5 of the algorithm.

We have shown that the slope of any point *x, y* within a region *[s, t]* bounded by vertical lines *s* and *s* + 1 and horizontal lines *t* and *t* + 1 can be easily computed. Thus the contour line (or isocost line) through *x, y* in region *[s, t]* may be readily drawn. Proceed- ing from one line in one region to the next line in the adjacent region until we come back to the starting point *(x, y)* then gives us a region of points in which any point has a total cost less than or equal to that of *(x, y)*.

Third, the lines V_0, V_{p+1} and H_0, H_{q+1} are required for defining the "exterior" regions. Although they are not included in the algorithm steps, the reader must take care to draw these lines.

Fourth, once we have determined the slopes of all the regions, the user may choose any point *(x, y)* other than a point that minimizes the objective function and draw a series of contour lines in order to get a region that contains points (i.e., facility locations) yielding as good or better objective function values than *(x, y)*. Thus step 6 could be repeated for several points to yield several such regions. Beginning with the innermost region, if any point in it is feasible, we use it as the optimal location. If not, we can go to the next innermost region to identify a feasible point. We repeat this procedure until we get a feasible point.

We now illustrate the contour line method with a numeric example.

Example 6

Consider Example 4. Suppose that the weight of facility 2 is not 10, but 20. Applying the median method, we can verify that the optimal location is (10, 10)—the centroid of department 2, where immovable structures exist. It is now desired to find a feasible and "near-optimal" location using the contour line method.

Solution

The contour line method is illustrated in the figure at the end of the solution.

Step 1 The vertical and horizontal lines V_1, V_2, V_3 and H_1, H_2, H_3, H_4 are drawn as shown. In addition to these lines, we draw lines V_0, V_4 and H_0, H_5 to identify the "exterior" regions.

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Step 2 The weights V_1, V_2, V_3, H_1, H_2, H_3, and H_4 are calculated by adding the weights of the points that fall on the respective lines. Note that for this example, $p = 3$ and $q = 4$.

Step 3 Because

$$\sum_{i=1}^{4} w_i = 38$$

Set $N_0 = D_0 = -38$

Step 4 Set

$N_1 = -38 + 2(8) = -22$

$N_2 = -22 + 2(26) = 30$

$N_3 = 30 + 2(4) = 38$

$D_1 = -38 + 2(6) = -26$

$D_2 = -26 + 2(4) = -18$

$D_3 = -18 + 2(8) = -2$

$D_4 = -2 + 2(20) = 3$

(These values are entered at the bottom of each column and to the left of each row in the following figure)

Step 5 Compute the slope of each region:

$S_{00} = -(-38/-38) = -1$ $S_{01} = -(-38/-26) = -1.46$ $S_{02} = -(-38/-18) = -2.11$

$S_{03} = -(-38/-2) = -19$ $S_{04} = -(-38/38) = 1$ $S_{10} = -(-22/-38) = -0.58$

$S_{11} = -(-22/-26) = -0.85$ $S_{12} = -(-22/-18) = -1.22$ $S_{13} = -(-22/-2) = -11$

$S_{14} = -(-22/38) = 0.58$ $S_{20} = -(30/-38) = 0.79$ $S_{21} = -(30/-26) = 1.15$

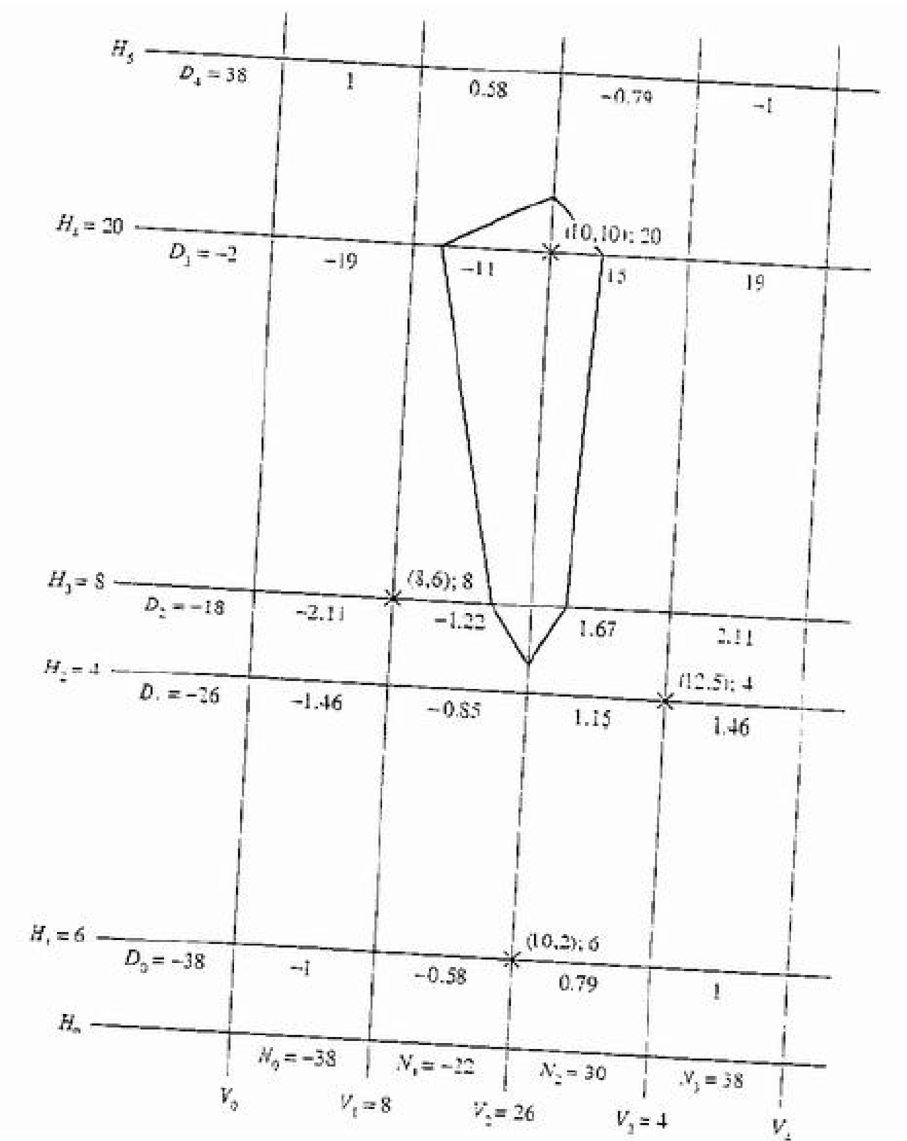
$S_{22} = -(30/-18) = 1.67$ $S_{23} = -(30/-2) = 15$ $S_{24} = -(30/38) = -0.79$

$S_{30} = -(38/-38) = 1$ $S_{31} = -(38/-26) = 1.46$ $S_{32} = -(38/18) = 2.11$

$S_{33} = -(38/-2) = 19$ $S_{34} = -(38/38) = -1$

(These slope values are shown inside each region.)

Step 6 When we draw contour lines through (9, 10), we get the region as shown in the following figure.



Because the copiers cannot be placed at the (10, 10) location, we drew contour lines through another nearby point, (9, 10). Locating the copiers anywhere possible within this region will give us a feasible, near-optimal solution.

In some location problems the distance function may not be linear but nonlinear. If it is quadratic, then determining the optimal location of the new facility is rather simple. To understand the method of solving such problems, consider the following objective function for single-facility location problems with a squared Euclidean distance metric:

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$$\text{Minimize} \quad TC = \sum_{i=1}^{m} c_i f_i \left[(x_i - \bar{x})^2 + (y_i - \bar{y})^2 \right] \tag{30}$$

As before, we substitute $w_i = c_i f_i$, where $i = 1, 2, \ldots, m$, and rewrite the objective function as:

$$\text{Minimize} \quad TC = \sum_{i=1}^{m} w_i (x_i - \bar{x})^2 + \sum_{i=1}^{m} w_i (y_i - \bar{y})^2 \tag{31}$$

Because this objective function can be shown to be convex, partially differentiating TC with respect to $\bar{x}$ and $\bar{y}$, setting the two resulting equations to zero and solving for $\bar{x}, \bar{y}$,provide the optimal location of the new facility:

$$\frac{\partial TC}{\partial \bar{x}} = 2 \sum_{i=1}^{m} w_i \bar{x} - 2 \sum_{i=1}^{m} w_i x_i = 0 \tag{32}$$

$$\therefore \quad \bar{x} = \frac{\sum_{i=1}^{m} w_i x_i}{\sum_{i=1}^{m} w_i} \tag{33}$$

$$\frac{\partial TC}{\partial \bar{y}} = 2 \sum_{i=1}^{m} w_i \bar{y} - 2 \sum_{i=1}^{m} w_i y_i = 0 \tag{34}$$

$$\therefore \quad \bar{y} = \frac{\sum_{i=1}^{m} w_i y_i}{\sum_{i=1}^{m} w_i} \tag{35}$$

It is easy to see that the optimal locations *x* and *y*are simply the weighted averages of the *x* and y coordinates of the existing facilities. This method of determining the optimal location is popularly known as the center-of-gravity or gravity or centroid method.

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If the optimal location determined by the gravity method is infeasible, we can again draw contour lines from neighboring points to find a feasible, near-optimal location. The contour lines will not be lines, however, but a circle through the point under consideration that has the optimal location as its center! [For a proof of this, see Francis and White (1974).] Thus, if the gravity method yields an optimal location (x,y) that is infeasible for the new facility, all we need to do is find any feasible point (x, y) that has the shortest Euclidean distance to (x,y) and locate the new facility at (x, y).

Example 7

Consider Example 4, suppose the distance metric to be used is squared Euclidean. Determine the optimal location of the new facility using the gravity method.

Solution

The $\sum_i w_i x_i$, $\sum_i w_i y_i$, and $\sum_i w_i$ values are calculated ing the table below. And using equations (33) and (35), we conclude that the normal coordinates ($\bar{x}$, $\bar{y}$) are

$\bar{x} = 272 / 28 = 9.7$
$\bar{y} = 180 / 28 = 6.4$

Department i	x_i	y_i	w_i	$w_i x_i$	$w_i y_i$
1	10	2	6	60	12
2	10	19	10	100	100
3	8	6	8	64	48
4	12	5	4	48	20
Total			**28**	**272**	**180**

If this location is not feasible, we find another feasible point that has the nearest euclidean distance to (9.7, 6.4) and that is a feasible location for the new facility.

Weiszfeld Method

The objective function for the single facility location problem with Euclidean distance can be written as:

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$$\text{Minimize} \qquad TC = \sum_{i=1}^{m} c_i f_i \left(\sqrt{(x_i - \bar{x})^2 + (y_i - \bar{y})^2} \right) \qquad (36)$$

As before substituting $w_i = c_i f_i$, taking the derivative of TC with respect to $\bar{x}, \bar{y}$, setting the deerivatives to zero, and solving for $\bar{x}, \bar{y}$ yield:

$$\frac{\partial TC}{\partial \bar{x}} = \frac{1}{2} \sum_{i=1}^{m} \frac{w_i [2(x_i - \bar{x})]}{\sqrt{(x_i - \bar{x})^2 + (y_i - \bar{y})^2}}$$

$$= \sum_{i=1}^{m} \frac{w_i x_i}{\sqrt{(x_i - \bar{x})^2 + (y_i - \bar{y})^2}} - \sum_{i=1}^{m} \frac{w_i \bar{x}}{\sqrt{(x_i - \bar{x}) + (y_i - \bar{y})^2}} = 0 \qquad (37)$$

$$\therefore \bar{x} = \frac{\left[\sum_{i=1}^{m} \frac{w_i x_i}{\sqrt{(x_i - \bar{x})^2 + (y_i - \bar{y})^2}} \right]}{\left[\sum_{i=1}^{m} \frac{w_i}{\sqrt{(x_i - \bar{x})^2 + (y_i - \bar{y})^2}} \right]} \qquad (38)$$

$$\frac{\partial TC}{\partial \bar{y}} = \frac{1}{2} \sum_{i=1}^{m} \frac{w_i [2(y_i - \bar{y})]}{\sqrt{(x_i - \bar{x})^2 + (y_i - \bar{y})^2}}$$

$$= \sum_{i=1}^{m} \frac{w_i y_i}{\sqrt{(x_i - \bar{x})^2 + (y_i - \bar{y})^2}} - \sum_{i=1}^{m} \frac{w_i \bar{y}}{\sqrt{(x_i - \bar{x})^2 + (y_i - \bar{y})^2}} = 0 \qquad (39)$$

$$\therefore \bar{y} = \frac{\left[\sum_{i=1}^{m} \frac{w_i y_i}{\sqrt{(x_i - \bar{x})^2 + (y_i - \bar{y})^2}} \right]}{\left[\sum_{i=1}^{m} \frac{w_i}{\sqrt{(x_i - \bar{x})^2 + (y_i - \bar{y})^2}} \right]} \qquad (40)$$

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Because $\sqrt{(x_i - \bar{x})^2 + (y_i - \bar{y})^2}$ appears twice in the denominators in Equations (38) and (40), the solution of $\bar{x}, \bar{y}$ is not defined when $\bar{x} = x_i$ and $\bar{y} = y_i$ for some i. This means that if the new facility's optimal coordinates coincide with those of an existing facility, Equations (38) and (40) are not defined and we therefore cannot use them in computing the opyimal coordinates $\bar{x}, \bar{y}$. The possibility of the optimal location of the new facility coinciding with that of an existing facility is very rare in practice, but it cannot be ruled out, so we need to devise another method for solving the single facility euclidean distance problem. Although (theoretically) optimal algorithms do not exist for this problem, a method from Weiszfeld (1936) is guaranteed to converge to the optimal location. This iterative algorithm is relatively straightforward.

Weiszfeld method

Step 0 Set iteration counter k=1:

$$\bar{x}^k = \frac{\sum_{i=1}^{m} w_i x_i}{\sum_{i=1}^{m} w_i} \qquad \bar{y}^k = \frac{\sum_{i=1}^{m} w_i y_i}{\sum_{i=1}^{m} w_i}$$

Step 1

$$\bar{x}^{k+1} = \frac{\sum_{i=1}^{m} \frac{w_i x_i}{\sqrt{(x_i - \bar{x}^k)^2 + (y_i - \bar{y}^k)^2}}}{\sum_{i=1}^{m} \frac{w_i}{\sqrt{(x_i - \bar{x}^k)^2 + (y_i - \bar{y}^k)^2}}}$$

$$\bar{y}^{k+1} = \frac{\sum_{i=1}^{m} \frac{w_i y_i}{\sqrt{(x_i - \bar{x}^k)^2 + (y_i - \bar{y}^k)^2}}}{\sum_{i=1}^{m} \frac{w_i}{\sqrt{(x_i - \bar{x}^k)^2 + (y_i - \bar{y}^k)^2}}}$$

Step 2

If $\bar{x}^{k-1} \simeq \bar{x}^k$ and $\bar{y}^{k+1} \simeq \bar{y}^k$, stop. Otherwise, set $k = k + 1$ and go to step 1.

Notice that the initial seeds for $\bar{x}$ and $\bar{y}$ were obtained from Equations (33) and (35), which were used in the gravity method. Although the Weiszfeld method is theoretically suboptimal, it provides $\bar{x}$ and $\bar{y}$ values that are very close to optimal. For practical purposes the algorithm works very well and can be readily implemented on a spreadsheet.

If the optimal location provided by the Weiszfeld method is not feasible, we can once again use the contour line method to draw contour lines and then choose a suitable, feasible, near-optimal location for the new facility. However, the methods for drawing the contour lines for the Euclidean distance metric, single-facility location problem are not exact. These approximate methods basically compute *TC* for a given point (x,y), choose a neighbouring x (or y) coordinate, and search for the y (or x) coordinate that yields the same *TC* value previously computed. This procedure is repeated until we come back to the starting point.

Example 8

Consider Example 6. Assume the distance metric is Euclidean and determine the optimal location of the new facility using the Weiszfeld method. Data for this problem are given in table below.

Department Number	x_i	y_i	w_i
1	10	2	6
2	10	10	20
3	8	6	8
4	12	5	4

Solution

The gravity method finds the initial seed (9.8, 7.4). With this as the starting solution, we apply step 1 of the Weiszfeld method repeatedly until two consecutive $\bar{x}$ and $\bar{y}$ values are equal. As shown in the following table, this occurs in the 25th iteration. For convenience, the total costs at the first 12, 20th and 25th iterations are also shown in the table. The optimal location for this problem, (10,10), is the same as that of an existing facility-department 2. This is no accident, it occurs because department 2's weight is more than half of the cumulative weights. In fact, when facility *i*'s weight is greater than or equal to half of the sum of the weights for all the remaining facilities, the new facility's optimal location will be the same as that of facility *i*. This is true under the rectilinear as well as Euclidean distance metrics. We therefore must use an approximate contour line methos to identify altenative, feasible solution.

Iteration Number	*x*	*y*	*TC*
1	9.7	7.8	113.4
2	9.7	8.2	111.9
3	9.8	8.4	110.8
4	9.8	8.7	109.9
5	9.8	8.9	109.1
6	9.9	9.0	108.5
7	9.9	9.2	108.0
8	9.9	9.3	107.6
9	9.9	9.4	107.2
10	9.9	9.5	106.9
11	9.9	9.6	106.7
12	10	9.6	106.5
⋮	⋮	⋮	⋮
20	10	9.9	105.6
⋮	⋮	⋮	⋮
25	10	10	105.5