ÇANKAYA UNIVERSITY Department of Industrial Engineering
2011-2012 SPRING SEMESTER

IE 302 FACILITIES DESIGN AND LOCATION COURSE READING NOTES PART 3

ADVANCED LOCATION MODELS

INTRODUCTION

Much of this chapter focuses on the more realistic, multifacility location problem. As discussed before, logistics management problems can be classified into:

- Location problems;
- Allocation problems; and
- Location-allocation problems

These are the five main issues in the more general location-allocation problem:
1. How many new **facilities are to** be located in the distribution network that consists of previously established facilities and customers?
2. Where should the new facilities be located?
3. How large should each new facility be? In other words, what is the capacity of the new facility?
4. How should customers be assigned to the new and existing facilities? More specifically, which facilities should be serving each customer?
5. Can more than one facility serve a customer?

A model that can answer all or most of these questions would be desirable, but we know by now that the more features we add to a model, the more difficult it is to solve. For the multifacility location problem, however, we do have a model that captures a variety of issues and considerations and yet is relatively easy to solve. Moreover, this model has been used by companies (e.g., Hunt-Wesson Foods, Inc.) to make logistical decisions. The algorithm to solve the model, however, is quite involved. It is based on Benders' decomposition approach. Before that we cover models for the location and allocation problems that are rather easy to solve.

LOCATION MODELS

Problems in which the new facilities have *no* interaction among themselves can be looked at as several *independent* single-facility location problems. For example, if we have to introduce three new facilities into an existing distribution network and there is no interaction among the three new facilities, then we can set up three independent single-facility location problems with the appropriate distance measure (rectilinear, squared Euclidean, or Euclidean),solve and simply combine the results to get a solution to the original problem. Although we can solve such special multifacility problems easily, if the location of one or more new facilities coincides with that of an existing one, finding optimal alternative feasible locations using the contour line method is extremely difficult for all but trivial two-facility problems.

Location problems in which there is interaction among new facilities and existing facilities and customers is more representative of the real world, so we now turn our attention to such problems.

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Multiple Facility Problems with Rectilinear Distances

Consider a distribution network with *m* facilities. It is desired to add *n* new facilities to the network. The coordinates of the *i*th existing facility are (a_i, b_i). The problem is to find coordinates of the *n* new facilities (x_i, y_i), where i = 1, 2,.. ., *n*, that minimize the total distribution cost. The "flow" from a new facility *i* to an existing facility *j* is denoted by g_{ij}, and that between new facilities *i* and *j* is f_{ij}. The cost per unit distance of travel between new facilities *i* and *j* is denoted as c_{ij} while that between new facility *i* and existing facility *j* is denoted as d_{ij}. This is the location problem:

Model 1

$$\text{Minimize} \quad \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij} f_{ij}\left[\,|x_i - x_j| + |y_i - y_j|\,\right] + \sum_{i=1}^{n}\sum_{j=1}^{m} d_{ij}\, g_{ij}\left[\,|x_i - a_j| + |y_i - b_j|\,\right] \qquad (1)$$

This nonlinear, unconstrained model can be transformed easily into an equivalent linear, constrained model. For example, we define

$$x_{ij}^{+} = \begin{cases} x_i - x_j & \text{if } (x_i - x_j) > 0 \\ 0 & \text{otherwise} \end{cases} \qquad (2)$$

$$x_{ij}^{-} = \begin{cases} x_j - x_i & \text{if } (x_i - x_j) \le 0 \\ 0 & \text{otherwise} \end{cases} \qquad (3)$$

We can observe that

$$|x_i - x_j| = x_{ij}^{+} + x_{ij}^{-} \qquad (4)$$

$$x_i - x_j = x_{ij}^{+} - x_{ij}^{-} \qquad (5)$$

A similar definition of y_{ij}^{+}, y_{ij}^{-}, xa_{ij}^{+}, xa_{ij}^{-}, yb_{ij}^{+}, and yb_{ij}^{-} yields

$$|y_i - y_j| = y_{ij}^{+} + y_{ij}^{-} \qquad (6)$$

$$y_i - y_j = y_{ij}^{+} - y_{ij}^{-} \qquad (7)$$

$$|x_i - a_j| = xa_{ij}^{+} + xa_{ij}^{-} \qquad (8)$$

$$x_i - a_j = xa_{ij}^{+} - xa_{ij}^{-} \qquad (9)$$

$$|y_i - b_j| = yb_{ij}^{+} + yb_{ij}^{-} \qquad (10)$$

$$y_i - b_j = yb_{ij}^{+} - yb_{ij}^{-} \qquad (11)$$

The transformed linear model is:

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$$\text{Minimize} \quad \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij} f_{ij} (x_{ij}^{+} + x_{ij}^{-} + y_{ij}^{+} - y_{ij}^{-}) + \sum_{i=1}^{n}\sum_{j=1}^{m} d_{ij} g_{ij} (xa_{ij}^{+} + xa_{ij}^{-} + yb_{ij}^{+} + yb_{ij}^{-}) \qquad (12)$$

Subject to constraints (5), (7), (9), (11)

$$x_{ij}^{+}, x_{ij}^{-}, y_{ij}^{+}, y_{ij}^{-} \geq 0 \qquad i, j = 1, 2, \ldots, n \qquad (13)$$

$$xa_{ij}^{+}, xa_{ij}^{-}, yb_{ij}^{+}, yb_{ij}^{-} \geq 0 \qquad i = 1, 2, \ldots, n, \quad j = 1, 2, \ldots, m \qquad (14)$$

$$x_i, y_i \text{ unrestricted in sign} \qquad i = 1, 2, \ldots, n \qquad (15)$$

For this model to be equivalent to expression (1), the solution must be such that either of the two new variables introduced, x_{ij}^{+} or x_{ij}^{-}, but not both, is greater than zero. [If both are, then the values of x_{ij}^{+} and x_{ij}^{-} do not satisfy their definitions in Equations (2) and (3).] Similarly, only one of the pairs y_{ij}^{+}, y_{ij}^{-} and xa_{ij}^{+}, xa_{ij}^{-} and yb_{ij}^{+}, yb_{ij}^{-} must be greater than zero. Recall that this condition had to be satisfied for the LMIP models as well as the median location model. Fortunately, they are automatically satisfied in the linear model presented here, just as they were in the median location model.

It turns out that the optimal *x* coordinate of each new facility is the same as that of an existing facility or customer. The same is true for the *y* coordinates. If it turns out that the *x* and *y* coordinates of a new facility coincide with the *x* and y coordinates of *a single* existing facility, we must find alternative feasible locations heuristically using rules of thumb—for example, locate a new facility in a feasible location that is within 5 miles of the optimal one. It is rather difficult to use the contour line methods that worked so well for the single-facility case,

Model 1 can be simplified by noting that x_i can be substituted as $a_j + xa_{ij}^{+} - xa_{ij}^{-}$ due to Equation (9) and the fact that x_i is unrestricted in sign. Similarly y_i may also be substituted, resulting in a model with 2*n* fewer constraints and variables than model 1.

Example 1

Tires and Brakes, Inc., is an automobile service company that specializes in tire and brake replacement. It has four service centers in a metropolitan area. It also has a warehouse that supplies tires, brakes, and other components to the service centers. The company manager has determined that he needs to add two more warehouses to improve component delivery service. At the same time he wants the location of the two new warehouses to minimize the cost of delivering components from the new warehouses to the existing facilities (four service centers and the existing warehouse) as well as between the new warehouses. The four service centers and warehouse have these coordinate locations: (8, 20), (8, 10), (10, 20), (16,30), and (35, 20). It is anticipated that there will be one trip per day between the new warehouses. The numbers of trips between the new warehouses (W_1, W_2) and the four service centers (SC_1 – SC_4) as well as the existing warehouse (SC_5) are provided in the matrix.

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$$\begin{array}{c} \\ W_1 \\ W_2 \end{array} \begin{array}{c} \begin{array}{ccccc} SC_1 & SC_2 & SC_3 & SC_4 & SC_5 \end{array} \\ \begin{bmatrix} 7 & 7 & 5 & 4 & 2 \\ 3 & 2 & 4 & 5 & 2 \end{bmatrix} \end{array}$$

Develop a model similar to the transformed model 1 to minimize the distribution cost and solve it using LINDO.

Solution

Because the cost per unit distance traveled is not given, we assume that the same type of vehicle is used for distribution and that the cost per unit distance traveled between any of the facilities is 1. Here are the model for the problem and the LINDO solution:

MIN XP12 + XN12 + YP12 + YN12 + XP21 + XN21 + YP21 + YN21 + 7XAP11 + 7XAN11 + 7YBP11 + 7YBN11 + 7XAP12 + 7XAN12 + 7YBP12 + 7YBN12 + 5XAP13 + 5XAN13 + 5YBP13 + 5YBN13 + 4XAP14 + 4XAN14 + 4YBP14 + 4YBN14 + 2XAP15 + 2XAN15 + 2YBP15 + 2YBN15 + 3XAP21 + 3XAN21 + 3YBP21 + 3YBN21 + 2XAP22 + 2XAN22 + 2YBP22 + 2YBN22 + 4XAP23 + 4XAN23 + 4YBP23 + 4YBN23 + 5XAP24 + 5XAN24 + 5YBP24 + 5YBN24 + 2XAP25 + 2XAN25 + 2YBP25 + 2YBN25

SUBJECT TO

2) – XP12 + XN12 + X1 – X2 = 0
3) – XP21 + XN21 – X1 + X2 = 0
4) – YP12 + YN12 – Y1 + Y2 = 0
5) – YP21 + YN21 – Y1 + Y2 = 0
6) – XAP11 + XAN11 + X1 = 8
7) – XAP12 + XAN12 + X1 = 8
8) – XAP13 + XAN13 + X1 = 10
9) – XAP14 + XAN14 + X1 = 16
10) – XAP15 + XAN15 + X1 = 35
11) – XAP21 + XAN21 + X2 = 8
12) – XAP22 + XAN22 + X2 = 8
13) – XAP23 + XAN23 + X2 = 10
14) – XAP24 + XAN24 + X2 = 16
15) – XAP25 + XAN25 + X2 = 35
16) – YBP11 + YBN11 + Y1 = 20
17) – YBP12 + YBN12 + Y1 = 10
18) – YBP13 + YBN13 + Y1 = 20
19) – YBP14 + YBN14 + Y1 = 30
20) – YBP15 + YBN15 + Y1 = 20
21) – YBP21 + YBN21 + Y2 = 20
22) – YBP22 + YBN22 + Y2 = 10
23) – YBP23 + YBN23 + Y2 = 20
24) – YBP24 + YBN24 + Y2 = 30
25) – YBP25 + YBN25 + Y2 = 20

END

FREE X1
FREE X2
FREE Y1
FREE Y2

LP OPTIMUM FOUND AT STEP 25

OBJECTIVE FUNCTION VALUE

1) 370.0000

VARIABLE	VALUE	REDUCED COST
XP12	.000000	2.000000
XN12	2.000000	.000000
YP12	.000000	2.000000
YN12	.000000	.000000
XP21	2.000000	.000000
XN21	.000000	2.000000
YP21	.000000	.000000
YN21	.000000	2.000000
XAP11	.000000	1.000000
XAN11	.000000	13.000000
YBP11	.000000	7.000000
YBN11	.000000	7.000000
XAP12	.000000	.000000
XAN12	.000000	14.000000
YBP12	10.000000	.000000
YBN12	.000000	14.000000
XAP13	.000000	10.000000
XAN13	2.000000	.000000
YBP13	.000000	4.000000
YBN13	.000000	6.000000
XAP14	.000000	8.000000
XAN14	8.000000	.000000
YBP14	.000000	8.000000
YBN14	10.000000	.000000
XAP15	.000000	4.000000
XAN15	27.000000	.000000
YBP15	.000000	4.000000
YBN15	.000000	.000000
XAP21	2.000000	.000000
XAN21	.000000	6.000000
YBP21	.000000	3.000000
YBN21	.000000	3.000000
XAP22	2.000000	.000000
XAN22	.000000	4.000000
YBP22	10.000000	.000000
YBN22	.000000	4.000000

XAP23	.000000	4.000000
XAN23	.000000	4.000000
YBP23	.000000	1.000000
YBN23	.000000	7.000000
XAP24	.000000	10.000000
XAN24	6.000000	.000000
YBP24	.000000	10.000000
YBN24	10.000000	.000000
XAP25	.000000	4.000000
XAN25	25.000000	.000000
YBP25	.000000	4.000000
YBN25	.000000	.000000
X1	8.000000	.000000
X2	10.000000	.000000
Y1	20.000000	.000000
Y2	20.000000	.000000

ROW	SLACK OR SURPLUS	DUAL PRICES
2)	.000000	-1.000000
3)	.000000	1.000000
4)	.000000	-1.000000
5)	.000000	1.000000
6)	.000000	6.000000
7)	.000000	7.000000
8)	.000000	-5.000000
9)	.000000	-4.000000
10)	.000000	-2.000000
11)	.000000	3.000000
12)	.000000	2.000000
13)	.000000	.000000
14)	.000000	-5.000000
15)	.000000	-2.000000
16)	.000000	.000000
17)	.000000	7.000000
18)	.000000	1.000000
19)	.000000	-4.000000
20)	.000000	-2.000000
21)	.000000	.000000
22)	.000000	2.000000
23)	.000000	3.000000
24)	.000000	-5.000000
25)	.000000	-2.000000

NO. ITERATIONS = 25

As mentioned, we could have reduced the problem size by substituting values for me free (i.e., unrestricted in sign) variables using some of the equality constraints. For example, we could have substituted XAP11 - XANI1 + 8 for XI (using row 6) or XAP15 - XAN15 + 35 (using row 10). This would have eliminated not only the corresponding row that was used for the substitution but also the free variable, thus reducing the problem size. In this example we made no substitution for any of the free variables and instead explicitly declared them.

In the solution to the model, notice that the location of each new facility coincides with that of an existing one. We find alternative feasible locations heuristically by choosing available locations close to the optimal ones for both new warehouses. Thus coordinate locations of (8.6, 20) and (9.3, 20) could be used for the TWO new warehouses. In fact, because the warehouses are so close together, the manager may even consider locating just one larger warehouse at coordinate location (9, 20) or reformulate the model under the assumption that only one new warehouse will be built and solve the resulting model to obtain the new location.

Multiple-Facility Problems with Euclidean Distances

Consider the following objective function for the Euclidean distance problem. (Recall that the notation was introduced earlier for the rectilinear distance problem.)

$$\text{Minimize} \quad \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} f_{ij} \left[\sqrt{(x_i - x_j)^2 + (y_i - y_j)^2} \right] + \sum_{i=1}^{n} \sum_{j=1}^{m} d_{ij}\, g_{ij} \left[\sqrt{(x_i - a_j)^2 + (y_i - b_j)^2} \right] \qquad (16)$$

As in the single-facility model, we can take the partial derivative of expression (16) with respect to me variables x_i and y_i, set the equations to zero, and solve for the variables because (16) can be shown to be a convex function. Taking the partial derivatives, we get

$$\sum_{j=1}^{n} \frac{c_{ij} f_{ij} (x_i - x_j)}{\sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}} + \sum_{j=1}^{m} \frac{d_{ij}\, g_{ij} (x_i - a_j)}{\sqrt{(x_i - a_j)^2 + (y_i - b_j)^2}} = 0 \quad i = 1, 2, \ldots, n \qquad (17)$$

$$\sum_{j=1}^{n} \frac{c_{ij} f_{ij} (y_i - y_j)}{\sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}} + \sum_{j=1}^{m} \frac{d_{ij}\, g_{ij} (y_i - b_j)}{\sqrt{(x_i - a_j)^2 + (y_i - b_j)^2}} = 0 \quad i = 1, 2, \ldots, n \qquad (18)$$

Because we have $2n$ variables and an equal number of constraints, we can solve Equations (17) and (18) to get the optimal (x, y) coordinates for all the n new facilities. As noted in the single-facility Euclidean distance model, however, we must be able to guarantee that the optimal location of any new facility does not coincide with that of any existing facility. Because the latter is not possible, we can develop an iterative heuristic procedure similar to what was done in the single-facility case. We add a small quantity ε to the denominator in each term on the left-hand side of Equations (17) and (18). Because Equations (17) and (18) are now defined even when the optimal location of a new facility coincides with that of an existing one, we can begin with an initial value for x_i, y_i for each new facility i and substitute these values into the following

Equations (19) and (20) to get the new values of x_i, y_i, (denoted as $x_i^{'}, y_i^{'}$, respectively). Notice that Equations (19) and (20) have been obtained by adding ε to the denominator of each term on the left-hand sides of Equations (17) and (18) and rewriting the equations:

$$x_i^{'} = \frac{\sum_{j=1}^{n} \frac{c_{ij} f_{ij} x_j}{\sqrt{(x_i - x_j)^2 + (y_i - y_j)^2 + \varepsilon}} + \sum_{j=1}^{m} \frac{d_{ij} g_{ij} a_j}{\sqrt{(x_i - a_j)^2 + (y_i - b_j)^2 + \varepsilon}}}{\sum_{j=1}^{n} \frac{c_{ij} f_{ij}}{\sqrt{(x_i - x_j)^2 + (y_i - y_j)^2 + \varepsilon}} + \sum_{j=1}^{m} \frac{d_{ij} g_{ij}}{\sqrt{(x_i - a_j)^2 + (y_i - b_j)^2 + \varepsilon}}} \quad i = 1, 2, \ldots, n \tag{19}$$

$$y_i^{'} = \frac{\sum_{j=1}^{n} \frac{c_{ij} f_{ij} y_j}{\sqrt{(x_i - x_j)^2 + (y_i - y_j)^2 + \varepsilon}} + \sum_{j=1}^{m} \frac{d_{ij} g_{ij} b_j}{\sqrt{(x_i - a_j)^2 + (y_i - b_j)^2 + \varepsilon}}}{\sum_{j=1}^{n} \frac{c_{ij} f_{ij}}{\sqrt{(x_i - x_j)^2 + (y_i - y_j)^2 + \varepsilon}} + \sum_{j=1}^{m} \frac{d_{ij} g_{ij}}{\sqrt{(x_i - a_j)^2 + (y_i - b_j)^2 + \varepsilon}}} \quad i = 1, 2, \ldots, n \tag{20}$$

The new values *of* x_i, y_i are substituted into the right-hand sides of Equations of (19) and (20) to get the next set of values. This procedure is continued until two successive x_i, y_i, values or the objective function values [obtained by substituting x_i, y_i, values in expression (16)] are nearly equal. Although it cannot be proved, we assume convergence has occurred at this point and stop. Upper and lower bounds on the optimal objective function value for the Euclidean distance problem can be found by looking at the rectilinear distance solution [see Francis and White (1974) and Pritsker and Ghare (1970) for more details]. Based on these bounds, we can tell how far off a given Euclidean solution is for a particular problem. For many practical problems, it has been found that the x_i, y_i, values for the new facilities determined via the iterative procedure are very close to optimal. The iterative procedure is rather easy to set up in a spreadsheet. Note that large values of ε will ensure a faster convergence, but the quality of the final solution is inferior compared with that obtained with a smaller ε value. Thus the user has to trade off quick convergence and solution quality and choose an appropriate value.

Example 2

Consider Example 1. Assume the Euclidean distance metric is more appropriate and that Tire and Brakes, Inc., does not currently have a warehouse. Determine where the two new warehouses are to be located.

Solution

Because there is no existing warehouse, we disregard that information in Example 1. A spreadsheet set up to iteratively calculate the x_i and y_i values is shown in Table 14.1. Also shown in the spreadsheet are the flow and ε values as well as the coordinate locations of the existing service centers. The columns labeled C_1 through C_4 give the values of the following part of Equation (19) calculated for each service center's co-ordinate location (a_j, b_j):

$$\frac{d_{ij}\, g_{ij}}{\sqrt{(x_i - a_j)^2 + (y_i - b_j)^2 + \varepsilon}}$$

Because this factor does not change for Equation (20), we do not show the values again in the y_i rows. The column labeled C_5 in the following table shows the values for the following part of Equations (19) and (20):

$$\frac{c_{ij}\, f_{ij}}{\sqrt{(x_i - x_j)^2 + (y_i - y_j)^2 + \varepsilon}}$$

Once again, because it is the same in both expressions, it is not shown in the y_i rows. Notice that in each iteration this value is the same for each x_i row because we have only two new warehouses to be located. The column labeled C_6 gives the sums of the values in columns C_1 through C_5 and is the denominator of Equations (19) and (20). Using an initial seed of (8,10) and (9,10) for the two facilities, we begin the iterative procedure. To determine the coordinates of the two new warehouses for the *k*th iteration, we use the ε, flow, (a_j, b_j) values, values in columns C_1 through C_6 for the previous (k - 1)th iteration, and Equations (19) and (20). This procedure is repeated until two successive x_i,y_i, values are equal. This occurs in the 13th iteration, and we therefore stop the procedure. (If we had used the total cost, shown in the last column as *TC*, to determine whether convergence had occurred, we would have stopped at the 12th iteration because solu-tions in this and the 11th yield the same total cost of 304.) If we had used large values of ε, convergence would have occurred much earlier, but then we may have obtained a solution inferior to the current one.

			SC_1	SC_2	SC_3	SC_4			
	W_1		7	7	5	4	Flow		
	W_2		3	2	4	5	values		
			A_1	a_2	a_3	a_4			
	x coordinate		8	8	10	16			
			B1	b2	b3	b4			
	y coordinate		20	10	20	30			
	ε		0.02						
Iteration	Coordinates		C_1	C_2	C_3	C_4	C_5	C_6	*TC*
1	x_1	8	0.6999	49.5	0.4902	0.183	0.99	52	*387*
	y_1	10							
	x_2	9	0.2985	1.98	0.398	0.236	0.99	3.9	
	y_2	10							
2	x_1	8.047	0.7216	20.89	0.5053	0.183	0.35	22.7	358
	y_1	10.3							
	x_2	8.941	0.4243	0.637	0.5644	0.272	0.35	2.25	
	y_2	12.994							
									321

3	x_1	8.11	0.7532	9.582	0.5273	0.19	0.15	11.2	
	y_1	10.708							
	x_2	9.615	0.9474	0.268	1.4541	0.351	0.15	3.17	
	y_2	17.28							
									297
4	x_1	8.231	0.8223	4.617	0.5753	0.194	0.11	6.32	
	y_1	11.492							
	x_2	9.88	1.5872	0.194	17.435	0.431	0.11	19.8	
	y_2	20.135							
									282
5	x_1	8.432	0.9772	2.425	0.683	0.197	0.13	4.42	
	y_1	12.851							
	x_2	9.95	1.5317	0.194	20.781	0.432	0.13	23.1	
	y_2	20.121							
									261
6	x_1	8.679	1.319	1.462	0.9214	0.199	0.18	4.08	
	y_1	14.738							
	x_2	9.962	1.5227	0.194	22.304	0.431	0.18	24.6	
	y_2	20.104							
									241
7	x_1	8.872	2.0341	1.04	1.4225	0.2	0.28	4.97	
	y_1	16.674							
	x_2	9.965	1.5207	0.194	22.866	0.431	0.28	25.3	
	y_2	20.097							
									227
8	x_1	8.942	3.3294	0.856	2.3171	0.199	0.45	7.15	
	y_1	18.125							
	x_2	9.966	1.52	0.194	23.056	0.431	0.45	25.7	
	y_2	20.095							
									221
9	x_1	8.93	5.0004	0.777	3.3398	0.197	0.65	9.96	
	y_1	19.351							
	x_2	9.967	1.5199	0.194	23.119	0.431	0.65	25.9	
	y_2	20.094							
									219
10	x_1	8.89	6.3056	7.45	3.8643	0.197	0.76	11.9	
	y_1	19.351							
	x_2	9.967	1.5198	0.194	23.14	0.431	0.76	26	
	y_2	20.094							
									218
11	x_1	8.41	7.70711	0.734	3.8643	0.197	0.78	12.7	
	y_1	19.497							
	x_2	9.967	1.5198	0.194	23.146	0.431	0.78	26.1	

	y_2	20.094							
12	x_1	8.794	7.5662	0.731	3.8568	0.197	0.77	13.1	218
	y_1	19.546							
	x_2	9.967	1.5198	0.194	23.149	0.431	0.77	26.1	
	y_2	20.094							
	x_1	8.754	7.9405	0.729	3.7693	0.197	0.75	13.4	218
	y_1	19.566							
	x_2	9.967	1.5198	0.194	23.15	0.431	0.75	26	
	y_2	20.094							

ALLOCATION MODEL

Manufacturing companies and some service organizations often find it necessary to maintain proximity to their markets and also to input sources. For manufacturing companies, the input sources may be raw materials, power, water, and so on. For service organizations, the input source may be a skilled labor pool—for example, companies such as Silicon Graphics specializing in computer software and hardware design. The allocation problem is then to find the quantity of raw material each supply source should be supplying to each plant, as well as the quantity of finished goods each plant: should be supplying to each customer. For the single-product case, this problem may be set up as a transportation model and hence may be solved rather easily (Das and Heragu 1988). This model is discussed in the next section.

Two-Stage Transportation Model

We consider an allocation model that has two stages of distribution. We formulate a linear programming (LP) model for this problem and show how a corresponding transportation tableau may be set up. The ideas are subsequently illustrated in a numeric example.

Consider this notation:

S_i capacity of supply source i, where i = 1. 2,..., p

P_j capacity of plant j where; = 1, 2,…,q

D_k demand at customer k, where k= 1, 2,…,q

c_{ij} cost of transporting one unit from supply source i to plant j

d_{jk} cost of transporting one unit from plant j to customer k

x_{jk} number of units of raw material shipped from supply source i to plant j

y_{jk} number of units of product shipped from plant j to customer k

This is the LP model:

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Model 2

Minimize
$$\sum_{i=1}^{p}\sum_{j=1}^{q} c_{ij}x_{ij} + \sum_{j=1}^{q}\sum_{k=1}^{r} d_{jk}y_{jk} \quad (21)$$

Subject to
$$\sum_{j=1}^{q} x_{ij} \le S_i \qquad i = 1, 2, \ldots, p \quad (22)$$

$$\sum_{i=1}^{p} x_{ij} \le P_j \qquad j = 1, 2, \ldots, q \quad (23)$$

$$\sum_{j=1}^{q} y_{jk} \ge D_k \qquad k = 1, 2, \ldots, r \quad (24)$$

$$\sum_{i=1}^{p} x_{ij} = \sum_{k=1}^{r} y_{jk} \qquad j = 1, 2, \ldots, q \quad (25)$$

$$x_{ij}, y_{jk} \ge 0 \qquad i = 1, 2, \ldots, p,\ j = 1, 2, \ldots, q,\ k = 1, 2, \ldots, r \quad (26)$$

The objective function (21) minimizes the cost of inbound as well as outbound shipments. Constraint (22) ensures that the raw material shipped out from each supply source does not exceed its capacity limits. Constraint (23) ensures that the raw material shipment received from all the supply sources at each plant does not exceed its capacity limits. Constraint (24) requires that the total amount of finished products shipped from the plants to each customer be sufficient to cover the demand. Constraint (25) is a material balance equation ensuring that all the raw material that comes into each plant is shipped out as finished product to customers. Notice that we are implicitly assuming that a unit of finished product requires one unit of raw material. If this is not the case, we can adjust the model easily, as discussed in Das and Heragu (1988).

For model 2 to be transformed into an equivalent transportation model, either the plants or the raw material supply sources (but not both) must have limited capacity. (Otherwise, the problem cannot be set up as a transportation model and hence we cannot use the well-known transportation algorithm. The problem may be formulated in model 2, however, and solved via the simplex algorithm.) Depending on whether supply sources or plants have limited capacities and whether supply exceeds demand, these four cases arise:

1. Supply source capacity is unlimited, plant capacity is limited, and total plant capacity is greater than total demand.
2. Supply source capacity is unlimited, plant capacity is limited, and total demand exceeds total plant capacity.
3. Plant capacity is unlimited, supply source capacity is limited, and total supply source capacity exceeds total demand.
4. Plant capacity is unlimited, supply source capacity is limited, and total demand exceeds total supply source capacity.

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In our discussion the supply sources are assumed to have unlimited capacities and the total plant capacity exceeds total demand (case 1). (Model 2 can be transformed rather easily into an equivalent transportation model for this problem.) The transportation tableau is set up in the following table. The third case is discussed in Example 3.

Unit transportation costs

Supply Source	Plant	Customer 1	2	...	r	Dummy Plant 1	2	...	q	Excess Plant Capacity	Capacity
1	1	c_{111}	c_{112}	...	c_{11r}	0	M	...	M	0	P_1
	2	c_{121}	c_{122}	...	c_{12r}	M	0	...	M	0	P_2
	⋮	⋱	⋱	⋱	⋱	⋮	⋮	⋱	⋮	⋮	⋮
	q	c_{1q1}	c_{1q2}	...	c_{1qr}	M	M	...	0	0	P_q
2	1	c_{211}	c_{212}	...	c_{21r}	0	M	...	M	0	P_1
	2	c_{221}	c_{222}	...	c_{22r}	M	0	...	M	0	P_2
	⋮	⋮	⋮	⋱	⋮	⋮	⋮	⋱	⋮	⋮	⋮
	q	c_{2q1}	c_{2q2}	...	c_{2qr}	M	M	...	0	0	P_q
p	1	c_{p11}	c_{p12}	...	c_{p1r}	0	M	...	M	0	P_1
	2	c_{p21}	c_{p22}	...	c_{p2r}	M	0	...	M	0	P_2
	⋮	⋮	⋮		⋮	⋮	⋮		⋮	⋮	⋮
	q	c_{pq1}	c_{pq2}	...	c_{pqr}	M	M	...	0	0	P_q
Demand		D_1	D_2	...	D_r	$(p-1)P_1$	$(p-1)P_2$	...	$(p-1)P_q$	$\sum_{j=1}^{q} P_j - \sum_{k=1}^{r} D_r$	$p \sum_{j=1}^{q} P_j$

In the transportation tableau, there are p rows corresponding to each plant even though only one plant can be set up at each location j, where $j = 1, 2 \ldots, q$. This accounts for the possibility that each plant may receive raw material from any supply source i. Because $p - 1$ excess rows have been introduced for each plant j with a capacity of P_j we need to remove this excess by introducing dummy plants 1, 2. .... q to absorb the excess plant capacity. The "demand" in these q columns is therefore $(p - 1)$ P_j, where; = 1, 2, ... q. The unit transportation costs for each of these columns are 0 in the corresponding rows and are large (denoted as M) in others. In other words, the dummy plant column j has 0 transportation costs in the jth row for each supply source i, where i = 1, 2, … p. The cost is M in all other rows for column j. The last demand column is introduced to absorb the excess of total plant capacity over total demand and has 0 cost in all the rows.

Now that the transportation tableau is set up, it can be solved efficiently with the transportation algorithm found in most elementary operations research textbooks (e.g.. Winston 1994) and in software packages such as STORM (Emmons et al. 1992) and QS (Chang and Sullivan 1991). Although the discussion thus far has pertained to problems with unlimited supply source capacities, in Example 3, we assume that plant capacities are unlimited in order to show the versatility of the approach.

Example 3

Two-stage distribution problem: RIFIN Co0mpany has recently developed a new method of manufacturing a type of chemical. The method involves refining a certain raw material that can be obtained from four overseas suppliers, A, B, C, and D, who have access to the four ports at Vancouver, Boston, Miami, and San

Francisco, respectively. RIFIN wants to determine the location for plants that will refine the material. Once refined, the chemical will bE transported via trucks to five outlets located in Dallas, Phoenix, Portland, Montreal, and Orlando. After an initial study, the choice of location for RIFIN's refineries has been narrowed down to Denver, Atlanta, and Pittsburgh. Assume that one unit of the raw material is required to make one unit of the chemical. The amount of raw material that can be obtained from suppliers A, B, C, and D and the amount of chemical required at the five outlets are given in the following table (a). The cost of transporting the raw material from each port to each potential refinery and the cost of trucking the chemical to outlets are provided in tables (b) and (c), respectively. Determine the locations of RIFIN's refining plants, the capacities at these plants, and the distribution pattern for the raw material and processed chemical.

(a) Supply and demand for four sources and five outlets

Raw Material Source	Supply	Outlet	Demand
A	1000	Dallas	900
B	800	Phoenix	800
C	800	Portland	600
D	700	Montreal	500
		Orlando	500

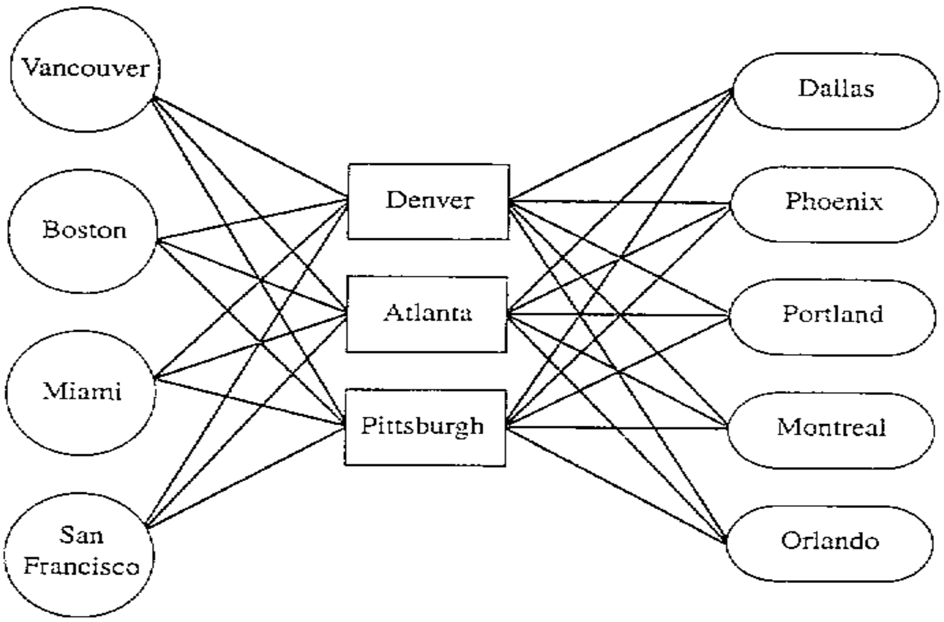
(b) Inland raw material transportation cost

From \ To	Denver	Atlanta	Pittsburgh
Vancouver	4	13	9
Boston	8	8	5
Miami	12	2	9
San Francisco	11	11	12

(c) Chemical trucking cost

From \ To	Dallas	Phoenix	Portland	Montreal	Orlando
Denver	28	26	12	30	30
Atlanta	10	22	23	29	8
Pittsburgh	18	21	23	18	21

Solution



Above figure is a pictorial representation of the RIFIN problem. We can reasonably assume that there is no practical limit on the capacity of the refineries at any of the three locations, Atlanta, Denver, and Pittsburgh, because the refineries have not been built yet. This assumption allows us to use the two-stage transportation method.

The transportation set-up is shown in the following table. Because we assume that a refinery capable of handling the total raw material supply can be built at each location, the supply rows of the transportation tableau are bounded by the capacity of each supply source. Also, because we are introducing more supply than is actually available at each source, we have to remove these excess units via the "dummy source" columns. Note that the cells that lie at the intersection of rows and column corresponding to *a* specific dummy supply source have zero costs in them and the rest are assigned a large positive value, M, to prohibit the use of these cells in the solution. For this problem, the total actual supply is equal to the total demand, and hence the excess source capacity column or the excess demand row is not needed.

The transportation problem may be solved to yield the solution (with a total cost of S65.400) in the following figure, which indicates that refineries should be built at all three locations.

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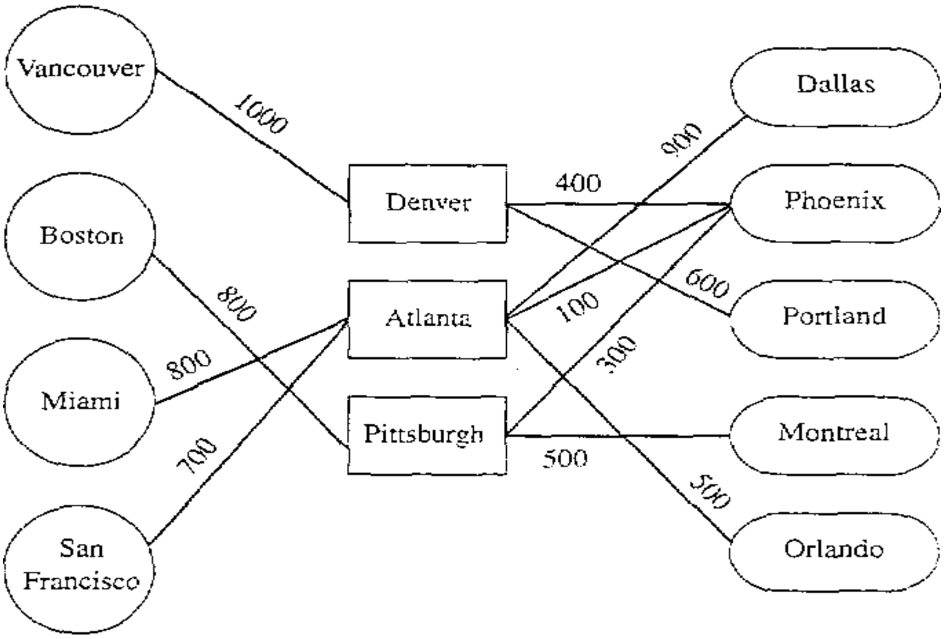

LOCATION-ALLOCATION MODELS

Generalized assignment problem, can be used to formulate location-allocation problems in which the objective is to determine the location of facilities to minimize the cost of assigning facilities to customers subject to the constraint that each facility be assigned to a prespecified number of customers. Similarly, the quadratic assignment model discussed in the context of a layout problem can be used at a macro level to determine the location of facilities given that these facilities have flow (interaction) among themselves. In this section we consider three other location-allocation models, each with specific applications:

1. Set covering model
2. Uncapacitated location-allocation model
3. Comprehensive location-allocation model

The models are discussed in order of the difficulty in solving them. For all the models, we present good heuristic or optimal solution procedures. The models determine the number of facilities to be located, where they are to be located, and the interaction between the facilities and customers. The first two are rather simple. The first considers only the cost of covering each customer with a facility. The second model considers a single product, one stage of distribution, facilities with unlimited capacity, and a customer to be served from several facilities. The third model relaxes several of these assumptions and therefore better represents the real-world location-allocation problem. To facilitate understanding of the third model, to provide a sound introduction, and to illustrate the use of efficient branch and bound algorithms, we begin our discussion of location-allocation problems with the first two simple models.

Set Covering Model

The set covering problem arises when it is necessary to ensure that each customer is covered by at least one service facility. For example, fire stations and other emergency facilities, libraries, community colleges, and state university campuses have to be located so that each population area or "customer" is within a certain range of distance from at least one facility. If a customer is within the desired range, we say the customer is covered. These are the parameters of the model:

c_j cost of locating facility at site *j*

$$a_{ij} = \begin{cases} 1 & \text{if facility located at site } j \text{ can cover customer } i \\ 0 & \text{otherwise} \end{cases}$$

$$x_j = \begin{cases} 1 & \text{if facility is located at site } j \\ 0 & \text{otherwise} \end{cases}$$

The set covering problem is given here:

Model 3

$$\text{Minimize} \quad \sum_{j=1}^{n} c_j x_j \qquad (27)$$

$$\text{Subject to} \quad \sum_{j=1}^{n} a_{ij} x_j \geq 1 \qquad i = 1, 2, \ldots, m \qquad (28)$$

$$x_j = 0 \text{ or } 1 \qquad j = 1, 2, \ldots, n \qquad (29)$$

In this 0-1 integer programming model, there are *m* customers and *n* facilities. Constraint (28) ensures that each customer is covered by at least one facility. The objective function (27) minimizes the cost of locating the required number of facilities. The model may be solved optimally using a general-purpose branch and bound technique, but that may be too time consuming for large problems. Hence the following greedy algorithm is used to obtain suboptimal solutions efficiently. It assumes that $c_j \geq 0, j = 1, 2. \ldots n$.

Example 4

A rural county administration wants to locate several medical emergency response units so that they can respond to any call in the county within 8 minutes. The county is divided into seven population zones. The distances between the centers of the zones are known and are given in the matrix in the following figure. The response units can be located in the centers of population zones 1-7 at a cost (in $10,000s) of 100, 80, 120, 110, 90, 90 and 110, respectively. Assuming the average travel speed during an emergency is 60 miles per hour, formulate an appropriate set covering model to determine where the units are to be located and how the population zones are to be covered. Solve the model using the greedy heuristic and calculate the solution cost.

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Distance between seven zones

$[d_{ij}] =$

	1	2	3	4	5	6	7
1	0	4	12	6	15	10	8
2	8	0	15	60	7	2	3
3	50	13	0	8	6	5	9
4	9	11	8	0	9	10	3
5	50	8	4	10	0	2	50
6	30	5	7	9	3	0	27
7	8	5	9	7	25	27	0

Solution

We define

$$a_{ij} = \begin{cases} 1 \text{ if zone } i\text{'s center can be reached from the center of zone } j \text{ within 8 minutes} \\ 0 \text{ otherwise} \end{cases}$$

and note that $d_{ij} > 8$, $d_{ij} \leq 8$ yield a_{ij} values of 0, 1 respectively. We can then set up the $[a_{ij}]$ matrix below:

Revised binary distance matrix

$[a_{ij}] =$

	1	2	3	4	5	6	7
1	1	1	0	1	0	0	1
2	1	1	0	0	1	1	1
3	0	0	1	1	1	1	0
4	0	0	1	1	0	0	1
5	0	1	1	0	1	1	0
6	0	1	1	0	1	1	0
7	1	1	0	1	0	0	1

The corresponding set covering model is:

$$\text{Minimize } 100x_1 + 80x_2 + 120x_3 + 110x_4 + 90x_5 + 90x_6 + 110x_7$$

$$\begin{array}{lrl}
\text{Subject to} & x_1 + x_2 + x_4 + x_7 & \geq 1 \\
& x_1 + x_2 + x_5 + x_6 + x_7 & \geq 1 \\
& x_3 + x_4 + x_5 + x_6 & \geq 1 \\
& x_3 + x_4 + x_7 & \geq 1 \\
& x_2 + x_3 + x_5 + x_6 & \geq 1 \\
& x_2 + x_3 + x_5 + x_6 & \geq 1 \\
& x_1 + x_2 + x_4 + x_7 & \geq 1 \\
& x_1, x_2, x_3, x_4, x_5, x_6, x_7 & = 0 \text{ or } 1
\end{array}$$

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Uncapacitated Location-Allocation Model

Consider this notation:

m	Number of potential facilities
n	Number of customers
c_{ij}	Cost of transporting one unit of product from facility I to customer j
F_i	Fixed cost of opening and operating facility i
D_j	Number of units demanded at customer j
x_{ij}	Number of units shipped from facility i to customer j
y_i	1 if facility is opened
	0 otherwise

The basic location-allocation model is given here:

Model 4

Minimize
$$\sum_{i=1}^{m} F_i y_i + \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} \qquad (30)$$

Subject to
$$\sum_{i=1}^{m} x_{ij} = D_j \qquad j = 1, 2, \ldots, n \qquad (31)$$

$$\sum_{j=1}^{n} x_{ij} \le y_i \sum_{j=1}^{n} D_j \qquad i = 1, 2, \ldots, m \qquad (32)$$

$$x_{ij} \ge 0 \qquad i = 1, 2, \ldots, m, \quad j = 1, 2, \ldots, n \qquad (33)$$

$$y_i = 0 \text{ or } 1 \qquad i = 1, 2, \ldots, m \qquad (34)$$

The objective function (30) minimizes the variable transportation cost as well as the fixed cost of opening and operating the facilities needed to support the distribution activities. Constraint (31) ensures that each of the n customers' demand is met fully by one or more of me m facilities. The objective function (30) and constraints (32) and (34) ensure that if a facility i ships goods to one or more customers, a corresponding fixed cost is incurred, and mat the total number of units shipped does not exceed the total demand at all the customers. On the other hand, if a facility does not ship goods to any customer, then no fixed cost is incurred. Constraint (33) is a nonnegativity constraint.

We now modify the formulation by making the following transformations of the x_{ij} variables and the c_{ij} parameter:

$$x'_{ij} = \frac{x_{ij}}{D_j} \quad c'_{ij} = c_{ij} D_j \quad \text{where } i = 1, 2, \ldots, m, \; j = 1, 2, \ldots, n \qquad (35)$$

The model 4 can be rewritten as follows:

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Model 5

$$\text{Minimize} \quad \sum_{i=1}^{m} F_i y_i + \sum_{i=1}^{m} \sum_{j=1}^{n} c'_{ij} x'_{ij} \tag{36}$$

$$\text{Subject to} \quad \sum_{i=1}^{m} x'_{ij} = 1 \qquad j = 1, 2, \ldots, n \tag{37}$$

$$\sum_{j=1}^{n} x'_{ij} \le n y_i \qquad i = 1, 2, \ldots, m \tag{38}$$

$$x'_{ij} \ge 0 \tag{39}$$

$$y_i = 0 \text{ or } 1 \tag{40}$$

Notice that x'_{ij} is the fraction of customer *j's* demand that is met by facility *i*. Ity can be seen that Expressions (36) and (37) are obtained by substituting $x'_{ij} = x_{ij} / D_j$, i= 1, 2, . . ., *m*, *j*= 1, 2, . . ., n. Constraint (38) is obtained as follows: Substituting $x'_{ij} = x_{ij} / D_j$ in constraint (32) we get

$$\sum_{j=1}^{n} x'_{ij} D_j \le y_i \sum_{j=1}^{n} D_j \qquad i = 1, 2, \ldots, m \tag{41}$$

Then dividing the left- and right-hand sides of (41) by $\sum D_j$, we get

$$\frac{1}{\sum_{j=1}^{n} D_j} \left[\sum_{j=1}^{n} x'_{ij} D_j \right] \le y_i \qquad i = 1, 2, \ldots m \tag{42}$$

Because the sum of the terms on the left-hand side of Expression (42) is less than or equal to y_i, each term must also be less than or equal to y_i, since x_{ij}, D_j are all greater than or equal to zero, i = 1. 2, ... , *m*, *j* = 1, 2,.. . , n. This gives us

$$\frac{x'_{ij} D_j}{\sum_{j=1}^{n} D_j} \le y_i \qquad j = 1, 2, \ldots, n \tag{43}$$

Because $D_j / \sum D_j$ is a positive fraction for each *j*, it follows that

$$x'_{ij} \le y_i \qquad j = 1, 2, \ldots, n \tag{44}$$

Adding the above *n* equations, we get

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$$\sum_{j=1}^{n} x'_{ij} \leq n y_i \qquad i = 1, 2, \ldots, m \tag{45}$$

Thus constraint (38) is equivalent to (32) m the following sense. Like (32), constraint (38) together with (40) and the objective function (36) ensures that if facility *i* serves any customer, then a corresponding fixed cost is incurred; otherwise, it is not. Thus model 5 is equivalent to model 4. Model 5 may be solved using the general-purpose branch and bound technique found in most introductory operations research textbooks (e.g., Winston 1994; Hillier and Lieberman 1995). This entails setting up a root node (i.e., a subproblem with model 5 without the integer restriction on the y_i, variables), solving this subproblem using the simplex algorithm, selecting a *y* variable— say, y_i—with a fractional value, branching on this variable, setting up two subproblems (nodes), one with a subproblem at the root node plus the constraint $y_i = 0$ and another with $y_i = 1$, solving the two subproblems (again using simplex), and deciding whether or not to prune a node based on these two tests:

1. The bound at the node is greater than or equal to the objective function value (OFV) of the best known feasible solution. (If no feasible solution has been identified yet, we proceed to test 2.)
2. The solution to the subproblem at the node is an all-integer (binary) solution. If a node passes either of the two tests, it is pruned and we update the best known OFV if necessary. Otherwise, we determine (arbitrarily or using specialized branching rules) the fractional y_i variable on which to branch, set up two additional subproblems (nodes), solve, and make pruning decisions as before. This procedure is repeated until all the nodes are pruned. At this point we have the optimal solution to the problem.

Although the general-purpose branch and bound technique can be applied to solve model 5, it is not very efficient because we have to solve several subproblems, one at each node, using the simplex algorithm. We now present a very efficient way of solving the subproblems that does not use the simplex algorithm. To facilitate our discussion, it is convenient to refer to x_{ij}, the fraction of customer *j*'s demand met by facility *i* in model 5, as simply x_{ij}. Thus x_{ij} in the remainder of this section does not refer to the number of units but rather to a fraction. Similarly c_{ij} now refers to c'_{ij} .

The central idea of the branch and bound algorithm is based on the following result: Suppose, at some stage of the branch and bound solution process, we are at a node where some facilities are closed (corresponding $y_i = 0$), some are open ($y_i = 1$), and the remaining are free; that is, a decision whether to open or close has not yet been made ($0 < y_i < 1$). We then define these parameters:

S_0 the set of facilities whose y_i value is equal to zero; $\{i: y_i = 0\}$

S_1 the set of facilities whose y_i value is equal to one; $\{i: y_i = 1\}$

S_2 the set of facilities whose y_i value is greater than zero but less than one $\{ i{:}0 < y_i < 1 \}$

Now examine the location-allocation model (36)-(39) for this node. It can be rewritten as model 6:

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Model 6

Minimize $$\sum_{i \in S_1} F_i + \sum_{i \in S_1} \sum_{j=1}^{n} c_{ij} x_{ij} + \sum_{i \in S_2} F_i y_i + \sum_{i \in S_2} \sum_{j=1}^{n} c_{ij} x_{ij} \quad (46)$$

Subject to $$\sum_{i=1}^{m} x_{ij} = 1 \qquad j = 1, 2, \ldots, n \quad (47)$$

$$\sum_{j=1}^{n} x_{ij} \le n y_i \qquad i = 1, 2, \ldots, m \quad (48)$$

$$x_{ij} \ge 0 \qquad i = 1, 2, \ldots, m, \ \ j = 1, 2, \ldots, n \quad (49)$$

Because x_{ij} is a fraction, it can be proved by contradiction that equality of expression (48) holds at optimality. From this, (48) can be written as

$$y_i = \frac{1}{n} \sum_{j=1}^{n} x_{ij} \quad (50)$$

Because the maximum value each x_{ij} can take is one, due to constraint (47), and the right-hand side of (50) is the sum of n x_{ij}'s divided by n, it is obvious that the maximum value that y_i can take is also one. Substituting the value of y_i, from (50) for i is an element of S_2 in (46), we get

Model 7

$$\text{Min} \sum_{i \in S_1} F_i + \left\{ \sum_{i \in S_1} \sum_{j=1}^{n} c_{ij} x_{ij} + \left[\sum_{i \in S_2} F_i \sum_{j=1}^{n} \frac{x_{ij}}{n} + \sum_{i \in S_2} \sum_{j=1}^{n} c_{ij} x_{ij} \right] \right\}$$

$$= \sum_{i \in S_1} F_i + \text{Min} \left\{ \sum_{i \in S_1} \sum_{j=1}^{n} c_{ij} x_{ij} + \left[\sum_{i \in S_2} \sum_{j=1}^{n} \left(c_{ij} + \frac{F_i}{n} \right) \left[x_{ij} \right] \right] \right\} \quad (51)$$

Model 7, which is equivalent to model 5 without the integer restrictions on the *y* variables, is a half assignment problem. It can be proved (again, by contradiction) that for each *j*= 1, 2, . . . ,*n* only *one* of x_{1j}, x_{2j}, …., x_{mj} will take on a value of one, due to constraint (47). In fact, for each *j*, the x_{ij} that takes on a value of one will be the one that has the smallest coefficient in Equation (51). Thus, in order to solve model 7, we only need to find for a specific y, the smallest coefficient of x_{ij} in Equation (51), *i* = 1, 2, . . . *m*, and set the corresponding x_{ij} equal to one and all other x_{ij}*'s* to zero. This is to be done for each *j* as shown next. We list the coefficients for each *j* as follows:

$$\begin{matrix} c_{ij} & \text{if } i \in S_1 \\ c_{ij} + \dfrac{F_i}{n} & \text{if } i \in S_2 \end{matrix} \qquad (52)$$

Select the smallest coefficient, and set the corresponding x_{ij} to one and all other x_{ij}'s. to zero. This method of determining the x_{ij}'s *is* called as the minimum coefficient rule. Notice that (52) does not include facility $i \in S_0$ because these are closed. Since the x_{ij}'s are known, the y_i values for $i \in S_2$ can be determined from Equation (50). Moreover, a lower bound on the partial solution of the node under consideration can be obtained via Equation (51) or simply by adding $\sum_{i \in S_0} F_i$ to the sum of the coefficients of the x_{ij} variables that have taken on a value of one (since all the other x_{ij}'s are equal to zero per the minimum coefficient rule). If it turns out that all the y_i values ($i \in S_2$) obtained from Equation (50) are binary, then we have a feasible solution and the lower bound obtained for the node from Equation (51) is also an upper bound for the original location-allocation problem. The node can therefore be pruned. If, on the other hand, one or more y_i, variables take on fractional values, then we need to branch on *one* of these variables, first by setting it equal to zero (and then to one), creating two corresponding nodes, updating S_0 or S_1 as appropriate, setting up model 6 for the nodes, and obtaining the solution and lower bound via the minimum coefficient rule discussed earlier, Equations (50) and (51). If the solution at a node has a lower bound greater than or equal to the best upper bound determined so far for the overall location-allocation problem, then it can be pruned because branching further on this node can only lead to worse solutions. We repeat the procedure of branching on nodes, solving the problem at each newly created node, determining the lower bound, and making pruning decisions until all the nodes are pruned. At that time, we have an optimal solution to the location-allocation model given by the node mat has a *feasible* solution with the least cost among all the nodes.

Comprehensive Location-Allocation Model

In all the models we have studied so far in this chapter and the preceding one, we did not explicitly consider multiple commodities. Now we present a comprehensive model that considers real-world factors and constraints. Consider this problem: Different types of products are produced at several plants that have known production capacities. The demand for each product type at each of several customer areas is also known. The products are shipped from plants to customer areas via intermediate warehouses with the restriction that each customer area be serviced by only one warehouse. This is done to improve customer service. Upper and lower bounds on the capacity of each warehouse, potential locations for the warehouses, inbound and outbound transportation costs at each of the warehouses (i.e., from each plant and to each customer area), and the fixed cost of opening and operating a warehouse at each potential location are known.

The problem is to find the locations for the warehouses, the corresponding capacities, the customers served by each warehouse, and how products are to be shipped from each plant to minimize the fixed and variable costs of opening and operating warehouses as well as the distribution costs. We use this notation:

S_{ij} Production capacity of product *i* at plant *j*

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D_{il}	Demand for product *I* at customer zone *l*
F_k	Fixed cost of operating warehouse *k*
V_{ik}	unit variable cost of handling product *i* at warehouse *k*
c_{ijkl}	Average unit cost of producing and transporting product *I* from plant *j* via warehouse *k* to customer area *l*
UC_k	Upper bound on capacity of warehouse *k*
LC_k	Lower bound on capacity of warehouse *k*
X_{ijkl}	Number of units of product *i* transported from plant *j* via warehouse *k* to customer area *l*
y_{kl}	1 if warehouse *k* serves customer area *l*; 0 otherwise
z_k	1 if warehouse is opened at location *k*; 0 otherwise

Here is the model for location-allocation:

Model 8

$$\text{Minimize} \quad \sum_{i=1}^{p}\sum_{j=1}^{q}\sum_{k=1}^{r}\sum_{l=1}^{s} c_{ijkl}\,x_{ijkl} + \sum_{i=1}^{p}\sum_{l=1}^{s} D_{il}\sum_{k=1}^{r} V_{ik}\,y_{kl} + \sum_{k=1}^{r} F_k z_k \qquad (53)$$

$$\text{Subject to} \quad \sum_{k=1}^{r}\sum_{l=1}^{s} x_{ijkl} \le S_{ij} \qquad i = 1, 2, \ldots, p, \quad j = 1, 2, \ldots, q \qquad (54)$$

$$\sum_{j=1}^{q} x_{ijkl} \ge D_{il}\,y_{kl} \qquad i = 1, 2, \ldots, p, \quad k = 1, 2, \ldots, r, \quad l = 1, 2, \ldots, s \qquad (55)$$

$$\sum_{k=1}^{r} y_{kl} = 1 \qquad l = 1, 2, \ldots, s \qquad (56)$$

$$\sum_{i=1}^{p}\sum_{l=1}^{s} D_{il}\,y_{kl} \ge LC_k\,z_k \qquad k = 1, 2, \ldots, r \qquad (57)$$

$$\sum_{i=1}^{p}\sum_{l=1}^{s} D_{il}\,y_{kl} \le UC_k\,z_k \qquad k = 1, 2, \ldots, r \qquad (58)$$

$$x_{ijkl} \ge 0 \qquad i = 1, 2, \ldots, p, \quad j = 1, 2, \ldots, q, \quad k = 1, 2, \ldots, r, \quad l = 1, 2, \ldots, s \qquad (59)$$

$$y_{kl}, z_k = 0 \text{ or } 1 \qquad k = 1, 2, \ldots, r, \quad l = 1, 2, \ldots, s \qquad (60)$$

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The objective function (53) of model 8 minimizes the inbound and outbound transportation costs as well as the production costs for each product at each warehouse. It also minimizes the fixed and variable costs of opening and operating the required number of warehouses. Constraint (54) ensures for each product that the capacity constraints at each plant are not violated. Constraint (55) ensures that the demand for each product at each customer zone is met. Constraints (56) and (60) require that each customer area be serviced by a single warehouse. Constraints (57) and (58) have a dual purpose. Not only do they enforce the upper and lower bounds on the warehouse capacity, but they also "connect" the y_{kl} and z_k variables. Because a warehouse can serve a customer area only if it is open, we must have $y_{kl} = 1$ when $z_k == 1$ and $y_{kl} = 0$ when $z_k = 0$ for each warehouse-customer area $\{k, l\}$ pair. These two conditions are satisfied by constraints (57) and (58), respectively.

We can easily add more linear constraints (not involving x_{ijkl} variables) to model 8 to:

- Impose upper and lower limits on the number of warehouses that can be opened;
- Enforce precedence relationships among warehouses (e.g., open warehouse at location 1 only if another is opened *at* location 3); and
- Enforce service constraints (e.g., if it is decided to open a certain warehouse, then a specific customer area must be served by it).

Other constraints that can be added are discussed further in Geoffrion and Graves (1974). Such constraints reduce the solution space, so they allow quicker solution of the model while giving the modeler much flexibility.

Model above can be solved using available mixed integer programming software, but due to the presence of binary integer variables y_{kl} and z_k, only small problems can be solved. Real world problems such as Hunt-Wesson Foods, Inc., location allocation problem considered in Geoffrion and Graves (1974), which had more than 11,000 constraints, 23,000 x_{ijkl} variables, and 700 y_{kl} and z_k binary variables, cannot be solved via general mixed integer programming algorithms. Such large problems have been rather easily solved using modified Bender's decomposition algorithm.